

Identities for the multiple zeta (star) values

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Abstract In this paper we prove some new identities for multiple zeta values and multiple zeta star values of arbitrary depth by using the methods of integral computations of logarithm function and iterated integral representations of series. By applying these formulas, we can prove that multiple zeta star values whose indices are the sequences $(\bar{1}, \{1\}_m, \bar{1})$ and $(2, \{1\}_m, \bar{1})$ can be expressed polynomially in terms of zeta values, polylogarithms and $\ln 2$. Finally, we also evaluate several restricted sum formulas involving multiple zeta values.

Keywords Multiple zeta value; multiple zeta star value; restricted sum formula.

AMS Subject Classifications (2010): 11M06; 11M40; 40B05; 33E20.

1 Introduction

The study of special values of multiple zeta functions and multiple zeta star functions concerns itself with relations between values at non-zero integer vectors $\mathbf{s} := (s_1, \dots, s_k)$ of sums of the form

$$\zeta(\mathbf{s}) \equiv \zeta(s_1, \dots, s_k) := \sum_{n_1 > \dots > n_k > 0} \prod_{j=1}^k n_j^{-|s_j|} \operatorname{sgn}(s_j)^{n_j}, \quad (1.1)$$

$$\zeta^*(\mathbf{s}) \equiv \zeta^*(s_1, \dots, s_k) := \sum_{n_1 \geq \dots \geq n_k \geq 1} \prod_{j=1}^k n_j^{-|s_j|} \operatorname{sgn}(s_j)^{n_j}, \quad (1.2)$$

commonly referred to as multiple zeta values and multiple zeta star values [2, 4, 5, 20, 23], respectively, where

$$\operatorname{sgn}(s_j) := \begin{cases} 1, & s_j > 0, \\ -1, & s_j < 0. \end{cases}$$

We are primarily interested in positive integer values of the arguments s_1, \dots, s_k , in which case it is seen that s_1 is both necessary for the sums (1.1) and (1.2) to converge. Of course, if $\sigma_1 = -1$, then we can allow $s_1 = 1$. Throughout the paper we will use \bar{p} to denote a negative entry $s_j = -p$. For example,

$$\zeta(\bar{s}_1, s_2) = \zeta(-s_1, s_2), \zeta^*(\bar{s}_1, s_2, \bar{s}_3) = \zeta^*(-s_1, s_2, -s_3).$$

We call $l(\mathbf{s}) := k$ the depth of (1.1), (1.2) and $|\mathbf{s}| := \sum_{j=1}^k |s_j|$ the weight. For convenience we set

$\zeta(\emptyset) = \zeta^*(\emptyset) = 1$ and $\{s_1, \dots, s_j\}_d$ the set formed by repeating the composition (s_1, \dots, s_j) d

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times.

A good deal of work on multiple zeta (star) values has focused on the problem of determining when ‘complicated’ sums can be expressed in terms of ‘simpler’ sums. Thus, researchers are interested in determining which sums can be expressed in terms of other sums of lesser depth. The origin of multiple zeta (star) values goes back to the correspondence of Euler with Goldbach in 1742-1743 (see [14]). Euler studied double zeta values and established some important relation formulas for them. For example, he proved that

$$2\zeta^*(k, 1) = (k+2)\zeta(k+1) - \sum_{i=1}^{k-2} \zeta(k-i)\zeta(i+1), \quad k \geq 2, \quad (1.3)$$

which, in particular, implies the simplest but nontrivial relation: $\zeta^*(2, 1) = 2\zeta(3)$ or equivalently $\zeta(2, 1) = \zeta(3)$. Moreover, he conjectured that the double zeta values would be reducible whenever weight is odd, and even gave what he hoped to be the general formula. The conjecture was first proved by David Borwein, Jonathan M. Borwein and Roland Girgensohn [3]. Some other interesting results of generalized double zeta values (also called Euler sums) see [1, 11].

The systematic study of multiple zeta values began in the early 1990s with the works of Hoffman [12], Zagier [24] and Borwein-Bradley-Broadhurst[2] and has continued with increasing attention in recent years (see [7–10]). The first systematic study of reductions up to depth 3 was carried out by Borwein and Girgensohn [6]. They also proved the conclusion: If $p+q+r$ is even or less than or equal to 10 or $p+q+r=12$, then triple zeta values $\zeta(q, p, r)$ (or $\zeta^*(q, p, r)$) can be expressed as a rational linear combination of products of zeta values and double zeta values. The set of the multiple zeta values has a rich algebraic structure given by the shuffle and the shuffle relations [12, 18, 25].

Additionally, it has been discovered in the course of the years that many multiple zeta (star) values admit expressions involving finitely the zeta values and polylogarithms, that are to say values of the Riemann zeta function and polylogarithm function,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1),$$

$$\text{Li}_s(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^s} \quad (\Re(s) \geq 1, x \in [-1, 1))$$

with positive integer arguments and $x = 1/2$. The relation between multiple zeta (star) values and the values of the Riemann zeta function and polylogarithm function has been studied by many authors, see [2, 4, 5, 15, 17, 20–23, 25]. For example, Zagier [25] proved that the multiple zeta star values $\zeta^*({2}_a, 3, {2}_b)$ and multiple zeta values $\zeta({2}_a, 3, {2}_b)$ are reducible to polynomials in zeta values, $a, b \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, and gave explicit formulae. Kh. Hesami Pilehrood et al.[21] and Zhonghua Li [17] provide two new proofs of Zagier’s formula for $\zeta^*({2}_a, 3, {2}_b)$ based on a finite identity for partial sums of the zeta-star series and hypergeometric series computations, respectively.

The purpose of the present is to establish some new family of identities for multiple integral representation of series. Then, we apply it to obtain a family of identities relating multiple zeta (star) values to zeta value and polylogarithms. Specially, we present some new recurrence relations for multiple zeta star values whose indices are the sequences $(\bar{1}, \{1\}_m, \bar{1})$, $(2, \{1\}_m, \bar{1})$, and prove that the multiple zeta values $\zeta(\bar{1}, \{1\}_{m-1}, \bar{1})$ and $\zeta(\bar{1}, \{1\}_{m-1}, \bar{1}, \{1\}_{k-1})$ can be expressed as a rational linear combination of zeta values, polylogarithms and $\ln 2$. Moreover, we also evaluate several restricted sum formulas involving multiple zeta values.

2 Evaluation of the multiple zeta star values $\zeta^*(\bar{1}, \{1\}_m, \bar{1})$ and $\zeta^*(2, \{1\}_m, \bar{1})$

In this section, we will show that the alternating multiple zeta star values $\zeta^*(\bar{1}, \{1\}_m, \bar{1})$ and $\zeta^*(2, \{1\}_m, \bar{1})$ satisfy certain recurrence relations that allow us to write them in terms of zeta values, polylogarithms and $\ln 2$. Now, we need three lemmas which will be useful in proving Theorem 2.4 and 2.5.

Lemma 2.1 *For $n, m \in \mathbb{N} := \{1, 2, \dots\}$ and $x \in [-1, 1)$. Then the following relation holds:*

$$\begin{aligned} \int_0^x t^{n-1} \ln^m(1-t) dt &= \frac{1}{n} \ln^m(1-x) (x^n - 1) + m! \frac{(-1)^m}{n} \sum_{1 \leq k_m \leq \dots \leq k_1 \leq n} \frac{x^{k_m}}{k_1 \cdots k_m} \\ &\quad - \frac{1}{n} \sum_{i=1}^{m-1} (-1)^{i-1} i! \binom{m}{i} \ln^{m-i}(1-x) \sum_{1 \leq k_i \leq \dots \leq k_1 \leq n} \frac{x^{k_i} - 1}{k_1 \cdots k_i}. \end{aligned} \quad (2.1)$$

Proof. The proof is by induction on m . Define $J(n, m; x) := \int_0^x t^{n-1} \ln^m(1-t) dt$, for $m = 1$, by a simple calculation, we can arrive at the conclusion that

$$J(n, 1; x) = \int_0^x t^{n-1} \ln(1-t) dt = \frac{1}{n} \left\{ x^n \ln(1-x) - \sum_{j=1}^n \frac{x^j}{j} - \ln(1-x) \right\},$$

and the formula is true. For $m > 1$ we proceed as follows. By using integration by parts we have the following recurrence relation

$$J(n, m; x) = \frac{1}{n} \ln^m(1-x) (x^n - 1) - \frac{m}{n} \sum_{k=1}^n J(k, m-1; x). \quad (2.2)$$

Suppose the lemma holds for $m-1$. Then the inductive hypothesis implies that the integral (2.2) is equal to

$$\begin{aligned} J(n, m; x) &= \frac{1}{n} \ln^m(1-x) (x^n - 1) + m! \frac{(-1)^m}{n} \sum_{j=1}^n \sum_{1 \leq k_{m-1} \leq \dots \leq k_1 \leq j} \frac{x^{k_{m-1}}}{k_1 \cdots k_{m-1}} \\ &\quad + \frac{m}{n} \sum_{i=1}^{m-2} (-1)^{i-1} i! \binom{m-1}{i} \ln^{m-1-i}(1-x) \sum_{1 \leq k_i \leq \dots \leq k_1 \leq j} \frac{x^{k_i} - 1}{k_1 \cdots k_i} \\ &\quad - \frac{m}{n} \ln^{m-1}(1-x) \sum_{j=1}^n \frac{x^j - 1}{j}. \end{aligned} \quad (2.3)$$

Thus, by a direct calculation, we deduce the desired result. This completes the proof of Lemma 2.1. \square

Lemma 2.2 ([22]) *For integer $m > 0$, the following identity holds,*

$$\int_0^1 \frac{\ln^m(1+x) \ln(1-x)}{1+x} dx = \frac{1}{m+1} \ln^{m+2} 2 - \zeta(2) \ln^m 2$$

$$- \sum_{k=1}^m \binom{m}{k} (-1)^{k+1} \left\{ \sum_{l=1}^k l! \binom{k}{l} (\ln 2)^{m-l} \text{Li}_{l+2} \left(\frac{1}{2} \right) - k! (\ln 2)^{m-k} \zeta(k+2) \right\}. \quad (2.4)$$

Lemma 2.3 ([22]) *For integer $m \geq 1$ and $z \in [0, 1]$. Then the following identity holds:*

$$\begin{aligned} \int_0^z \frac{\ln^m(1+x)}{x} dx &= \frac{1}{m+1} \ln^{m+1}(1+z) + m! \left(\zeta(m+1) - \text{Li}_{m+1} \left(\frac{1}{1+z} \right) \right) \\ &\quad - m! \sum_{j=1}^m \frac{\ln^{m-j+1}(1+z)}{(m-j+1)!} \text{Li}_j \left(\frac{1}{1+z} \right). \end{aligned} \quad (2.5)$$

Now the main results in this section are the following theorems.

Theorem 2.4 *For integer $m > 1$, we have the recurrence relation*

$$\begin{aligned} \zeta^*(\bar{1}, \{1\}_{m-1}, \bar{1}) &= \frac{(-1)^{m-1}}{(m-1)!} \zeta(2) \ln^{m-1} 2 \\ &\quad - \frac{(-1)^{m-1}}{m!} \sum_{i=1}^{m-1} (-1)^{i+1} i! \binom{m}{i} (\ln 2)^{m-i} \{ \zeta^*(\bar{1}, \{1\}_{i-1}, \bar{1}) - \zeta^*(\bar{1}, \{1\}_i) \} \\ &\quad + \frac{(-1)^{m-1}}{(m-1)!} \sum_{k=1}^{m-1} \binom{m-1}{k} (-1)^{k+1} \left\{ \sum_{l=1}^k l! \binom{k}{l} (\ln 2)^{m-1-l} \text{Li}_{l+2} \left(\frac{1}{2} \right) - k! (\ln 2)^{m-1-k} \zeta(k+2) \right\}. \end{aligned} \quad (2.6)$$

Proof. Multiplying (2.1) by $(-1)^{n-1}$ and summing with respect to n , the result is

$$\begin{aligned} \int_0^x \frac{\ln^m(1-t)}{1+t} dt &= \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^x t^{n-1} \ln^m(1-t) dt \\ &= \ln^m(1-x) \sum_{n=1}^{\infty} \frac{x^n - 1}{n} (-1)^{n-1} + m! (-1)^m \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{1 \leq k_m \leq \dots \leq k_1 \leq n} \frac{x^{k_m}}{k_1 \dots k_m} \\ &\quad - \sum_{i=1}^{m-1} (-1)^{i-1} i! \binom{m}{i} \ln^{m-i}(1-x) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{1 \leq k_i \leq \dots \leq k_1 \leq n} \frac{x^{k_i} - 1}{k_1 \dots k_i}. \end{aligned} \quad (2.7)$$

On the other hand, by integration by parts, we obtain the formula

$$\begin{aligned} &\lim_{x \rightarrow -1} \left\{ \int_0^x \frac{\ln^m(1-t)}{1+t} dt - \ln^m(1-x) \ln(1+x) \right\} \\ &= \lim_{x \rightarrow -1} \left\{ m \int_0^x \frac{\ln^{m-1}(1-t) \ln(1+t)}{1-t} dt \right\} \\ &= m \int_0^{-1} \frac{\ln^{m-1}(1-t) \ln(1+t)}{1-t} dt \end{aligned}$$

$$= -m \int_0^1 \frac{\ln^{m-1}(1+t) \ln(1-t)}{1+t} dt,$$

Hence, letting x approach -1 in (2.7) and combining (2.4), we deduce (2.6). \square

Noting that, taking $x \rightarrow 1$ in (2.7), we have the result

$$\int_0^1 \frac{\ln^m(1-t)}{1+t} dt = m!(-1)^{m-1} \zeta^*(\bar{1}, \{1\}_m) = (-1)^m m! \text{Li}_{m+1}\left(\frac{1}{2}\right).$$

Therefore, we conclude that

$$\zeta^*(\bar{1}, \{1\}_m) = -\text{Li}_{m+1}\left(\frac{1}{2}\right). \quad (2.8)$$

Theorem 2.5 *For integer $m > 1$, we have the recurrence relation*

$$\begin{aligned} \zeta^*(2, \{1\}_{m-1}, \bar{1}) &= \frac{m+1}{(m+2)!} (-1)^{m+1} \ln^{m+2} 2 + (m+1) (-1)^{m+1} \left(\zeta(m+2) - \text{Li}_{m+2}\left(\frac{1}{2}\right) \right) \\ &\quad - (m+1) (-1)^{m+1} \sum_{j=1}^{m+1} \frac{(\ln 2)^{m+2-j}}{(m+2-j)!} \text{Li}_j\left(\frac{1}{2}\right) - \frac{3}{2} \frac{(-1)^{m+1}}{m!} (\ln 2)^m \zeta(2) \\ &\quad - \frac{(-1)^{m+1}}{m!} \sum_{i=1}^{m-1} (-1)^{i-1} i! \binom{m}{i} (\ln 2)^{m-i} \{ \zeta^*(2, \{1\}_{i-1}, \bar{1}) - \zeta^*(2, \{1\}_i) \}. \end{aligned} \quad (2.9)$$

Proof. Similarly as in the proof of Theorem 2.4, we consider the integral

$$\int_0^{-1} \frac{\ln^{m+1}(1-t)}{t} dt = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{-1} t^{n-1} \ln^m(1-t) dt. \quad (2.10)$$

Putting $x \rightarrow -1$ in (2.1), we deduce that

$$\begin{aligned} \int_0^{-1} t^{n-1} \ln^m(1-t) dt &= \frac{1}{n} \ln^m 2 ((-1)^n - 1) + m! \frac{(-1)^m}{n} \zeta_n^*(\{1\}_m, \bar{1}) \\ &\quad - \frac{1}{n} \sum_{i=1}^{m-1} (-1)^{i-1} i! \binom{m}{i} \ln^{m-i} 2 \{ \zeta_n^*(\{1\}_{i-1}, \bar{1}) - \zeta_n^*(\{1\}_i) \}. \end{aligned} \quad (2.11)$$

Setting $x \rightarrow 1$ in (2.5) and combining (2.10) with (2.11), we obtain (2.9). \square

By considering the case $m = 2$ in (2.6) and (2.9) we get

$$\begin{aligned} \zeta^*(\bar{1}, 1, \bar{1}) &= \frac{1}{8} \zeta(3) + \frac{1}{2} \zeta(2) \ln 2 - \frac{1}{6} \ln^3 2, \\ \zeta^*(2, 1, \bar{1}) &= \frac{1}{8} \ln^4 2 + 3 \text{Li}_4\left(\frac{1}{2}\right) - 3 \zeta(4) - \frac{3}{2} \zeta(2) \ln^2 2 + \frac{7}{8} \zeta(3) \ln 2. \end{aligned}$$

Therefore, from Theorem 2.4 and Theorem 2.5, we know that the alternating multiple zeta star values $\zeta^*(\bar{1}, \{1\}_m, \bar{1})$ and $\zeta^*(2, \{1\}_m, \bar{1})$ can be expressed as a rational linear combination of zeta values, polylogarithms and $\ln 2$.

3 Some results on multiple zeta values

In this section, we use certain multiple integral representations to evaluate several multiple zeta values. We need the following lemma.

Lemma 3.1 ([22]) *For integer $k > 0$ and $x \in [-1, 1)$, we have that*

$$\ln^k(1-x) = (-1)^k k! \sum_{n=1}^{\infty} \frac{x^n}{n} \zeta_{n-1}(\{1\}_{k-1}), \quad (3.1)$$

$$s(n, k) = (n-1)! \zeta_{n-1}(\{1\}_{k-1}). \quad (3.2)$$

where $s(n, k)$ is called (unsigned) Stirling number of the first kind (see [16]). The Stirling numbers $s(n, k)$ of the first kind satisfy a recurrence relation in the form

$$s(n, k) = s(n-1, k-1) + (n-1)s(n-1, k), \quad n, k \in \mathbb{N},$$

with $s(n, k) = 0, n < k, s(n, 0) = s(0, k) = 0, s(0, 0) = 1$.

Here the multiple harmonic number (also called the partial sums of multiple zeta value) and multiple harmonic star number (also called the partial sums of multiple zeta star value) are defined by

$$\zeta_n(s_1, s_2, \dots, s_k) := \sum_{n \geq n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_k^{s_k}},$$

$$\zeta_n^*(s_1, s_2, \dots, s_k) := \sum_{n \geq n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_k^{s_k}},$$

when $n < k$, then $\zeta_n(s_1, s_2, \dots, s_k) = 0$, and $\zeta_n(\emptyset) = \zeta_n^*(\emptyset) = 1$.

The main results are the following theorems and corollary.

Theorem 3.2 *For integers $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$, then*

$$\zeta(\bar{1}, \{1\}_m, \bar{1}, \{1\}_{k-1}) = (-1)^{m+1} \text{Li}_{k+1, \{1\}_m} \left(\frac{1}{2} \right), \quad (3.3)$$

where the multiple polylogarithm function

$$\text{Li}_{s_1, s_2, \dots, s_k}(x) := \sum_{n_1 > \dots > n_k > 0} \frac{x^{n_1}}{n_1^{s_1} \dots n_k^{s_k}} \quad (3.4)$$

is defined for positive integers s_j , and x is a real number satisfying $0 \leq x < 1$. Of course, if $s_1 > 1$, then we can allow $x = 1$.

Proof. To prove the identity (3.3), we consider the following multiple integral

$$M_m(k) := \int_0^1 \frac{1}{1+t_1} dt_1 \dots \int_0^{t_{m-1}} \frac{1}{1+t_m} dt_m \int_0^{t_m} \frac{\ln^k(1-t_{m+1})}{1+t_{m+1}} dt_{m+1}. \quad (3.5)$$

By using power series expansion and formula (3.1), we deduce that

$$M_m(k) = (-1)^{k+m+1} k! \sum_{n_1, n_2, \dots, n_{m+1}=1}^{\infty} (-1)^{n_1 + \dots + n_{m+1}} \frac{\zeta_{n_1-1}(\bar{1}, \{1\}_{k-1})}{n_1}$$

$$\begin{aligned}
& \times \int_0^1 t_1^{n_{m+1}-1} dt_1 \cdots \int_0^{t_{m-1}} t_m^{n_1+n_2-1} dt_m \\
& = (-1)^{k+m+1} k! \sum_{n_1, n_2, \dots, n_{m+1}=1}^{\infty} (-1)^{n_1+\dots+n_{m+1}} \frac{\zeta_{n_1-1}(\bar{1}, \{1\}_{k-1})}{n_1(n_1+n_2)\cdots(n_1+\dots+n_{m+1})} \\
& = (-1)^{k+m+1} k! \sum_{n_1 > \dots > n_{m+1} \geq 1}^{\infty} \frac{\zeta_{n_{m+1}-1}(\bar{1}, \{1\}_{k-1})}{n_1 n_2 \cdots n_{m+1}} (-1)^{n_1} \\
& = (-1)^{k+m+1} k! \zeta(\bar{1}, \{1\}_m, \bar{1}, \{1\}_{k-1}). \tag{3.6}
\end{aligned}$$

Hence, $M_{m-1}(k) = (-1)^{k+m} k! \zeta(\bar{1}, \{1\}_{m-1}, \bar{1}, \{1\}_{k-1})$. On the other hand, applying the change of variables

$$t_i \mapsto 1 - t_{m+2-i}, \quad i = 1, 2, \dots, m+1.$$

to the above multiple integral $M_m(k)$, we have

$$\begin{aligned}
M_m(k) &= \int_0^1 \frac{\ln^k(t_1)}{2-t_1} dt_1 \int_0^{t_1} \frac{1}{2-t_2} dt_2 \cdots \int_0^{t_m} \frac{1}{2-t_{m+1}} dt_{m+1} \\
&= \sum_{n_1, \dots, n_{m+1}=1}^{\infty} \frac{1}{2^{n_1+\dots+n_{m+1}}} \int_0^1 t_1^{n_{m+1}-1} \ln^k(t_1) dt_1 \int_0^{t_1} t_2^{n_m-1} dt_2 \cdots \int_0^{t_m} t_{m+1}^{n_1-1} dt_{m+1} \\
&= \sum_{n_1, \dots, n_{m+1}=1}^{\infty} \frac{1}{2^{n_1+\dots+n_{m+1}}} \cdot \frac{1}{n_1(n_1+n_2)\cdots(n_1+\dots+n_m)} \int_0^1 t_1^{n_1+\dots+n_{m+1}-1} \ln^k(t_1) dt_1 \\
&= k! (-1)^k \sum_{n_1, \dots, n_{m+1}=1}^{\infty} \frac{1}{2^{n_1+\dots+n_{m+1}}} \cdot \frac{1}{n_1(n_1+n_2)\cdots(n_1+\dots+n_m)(n_1+\dots+n_{m+1})^{k+1}} \\
&= k! (-1)^k \sum_{1 \leq n_{m+1} < \dots < n_1}^{\infty} \frac{1}{n_{m+1} \cdots n_2 n_1^{k+1} 2^{n_1}} \\
&= k! (-1)^k \text{Li}_{k+1, \{1\}_m} \left(\frac{1}{2} \right). \tag{3.7}
\end{aligned}$$

Therefore, combining (3.6) and (3.7) we may deduce the result. \square

Theorem 3.3 For integers $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$, then

$$\begin{aligned}
\zeta(\bar{1}, \{1\}_m, \bar{1}, \{1\}_{k-1}) &= \frac{(-1)^{m+k}}{(k-1)!} \sum_{j=0}^{k-1} (-1)^j (\ln 2)^{k-1-j} j! \binom{k-1}{j} \\
&\quad \times \left\{ \zeta\left(m+2, \{1\}_j\right) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m+1-l)!} \text{Li}_{l+1, \{1\}_j} \left(\frac{1}{2} \right) \right\}. \tag{3.8}
\end{aligned}$$

Proof. By using (3.1), we can find that

$$\int_0^1 \frac{(\ln x)^k \ln^{m+1} \left(1 - \frac{x}{2}\right)}{x} dx = (-1)^{m+1} (m+1)! \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(\{1\}_m)}{n 2^n} \int_0^1 x^{n-1} \ln^k x dx$$

$$\begin{aligned}
&= (-1)^{m+k+1} (m+1)!k! \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(\{1\}_m)}{n^{k+2}2^n} \\
&= (-1)^{m+k+1} (m+1)!k! \text{Li}_{k+2, \{1\}_m} \left(\frac{1}{2} \right). \tag{3.9}
\end{aligned}$$

Then it is readily seen that

$$\text{Li}_{k+1, \{1\}_m} \left(\frac{1}{2} \right) = \frac{(-1)^{m+k}}{(m+1)!(k-1)!} \int_0^1 \frac{(\ln x)^{k-1} \ln^{m+1} \left(1 - \frac{x}{2} \right)}{x} dx, \tag{3.10}$$

$$\zeta(\bar{1}, \{1\}_m, \bar{1}, \{1\}_{k-1}) = \frac{(-1)^{k+1}}{(m+1)!(k-1)!} \int_0^1 \frac{(\ln x)^{k-1} \ln^{m+1} \left(1 - \frac{x}{2} \right)}{x} dx. \tag{3.11}$$

On the other hand, changing $x = 2(1-u)$ in above integral, we conclude that

$$\begin{aligned}
&\int_0^1 \frac{(\ln x)^{k-1} \ln^{m+1} \left(1 - \frac{x}{2} \right)}{x} dx \\
&= \int_{\frac{1}{2}}^1 \frac{(\ln 2 + \ln(1-u))^{k-1} \ln^{m+1} u}{1-u} du \\
&= \sum_{j=0}^{k-1} (\ln 2)^{k-1-j} \binom{k-1}{j} \int_{\frac{1}{2}}^1 \frac{\ln^j(1-u) \ln^{m+1} u}{1-u} du \\
&= \sum_{j=0}^{k-1} (-1)^j (\ln 2)^{k-1-j} j! \binom{k-1}{j} \sum_{n=1}^{\infty} \zeta_{n-1}(\{1\}_j) \int_{\frac{1}{2}}^1 u^{n-1} \ln^{m+1} u du \\
&= (m+1)!(-1)^{m+1} \sum_{j=0}^{k-1} (-1)^j (\ln 2)^{k-1-j} j! \binom{k-1}{j} \\
&\quad \times \left\{ \zeta(m+2, \{1\}_j) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m+1-l)!} \text{Li}_{l+1, \{1\}_j} \left(\frac{1}{2} \right) \right\}. \tag{3.12}
\end{aligned}$$

Thus, substituting (3.12) into (3.11), we obtain the desired result. The proof of Theorem 3.3 is finished. \square

From Theorem 3.2 and Theorem 3.3, we immediately derive the following special case of the multiple zeta values.

Corollary 3.4 (Conjectured in [2]) *For integer $m \in \mathbb{N}_0$, then the following identities hold:*

$$\text{Li}_{2, \{1\}_m} \left(\frac{1}{2} \right) = \zeta(m+2) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m+1-l)!} \text{Li}_{l+1} \left(\frac{1}{2} \right), \tag{3.13}$$

$$\zeta(\bar{1}, \{1\}_m, \bar{1}) = (-1)^{m+1} \left\{ \zeta(m+2) - \sum_{l=0}^{m+1} \frac{(\ln 2)^{m+1-l}}{(m+1-l)!} \text{Li}_{l+1}\left(\frac{1}{2}\right) \right\}. \quad (3.14)$$

Proof. Setting $k = 1$ in (3.3) and (3.8) we may easily deduce the results. \square

Note that Corollary 3.4 is an immediate corollary of Zlobin's Theorem 9 (see [26]).

Theorem 3.5 *For positive integers m and k , we have*

$$\begin{aligned} \zeta(\bar{1}, \{1\}_{m-1}, \bar{1}, \bar{1}, \{1\}_{k-1}) &= \frac{(-1)^{k-1}}{m!k!} \{k(\ln 2)^m I(k-1) - (m+k)I(m+k-1)\} \\ &\quad - \sum_{i=1}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, \bar{1}, \bar{1}, \{1\}_{k-1}), \end{aligned} \quad (3.15)$$

where $I(k)$ denotes the integral on the left hand side of (2.4), namely

$$I(k) := \int_0^1 \frac{\ln^k(1+x) \ln(1-x)}{1+x} dx.$$

Proof. By a similar argument as in the proof of Theorem 2.10, we consider the following multiple integral

$$J_m(k) := \int_0^1 \frac{1}{1+t_1} dt_1 \cdots \int_0^{t_{m-2}} \frac{1}{1+t_{m-1}} dt_{m-1} \int_0^{t_{m-1}} \frac{1}{1+t_m} dt_m \int_0^{t_m} \frac{\ln^k(1+t_{m+1})}{1-t_{m+1}} dt_{m+1}. \quad (3.16)$$

By using (3.1), we deduce that

$$\int_0^x \frac{1}{1+t} dt \int_0^t \frac{\ln^k(1+u)}{1-u} du = (-1)^{k-1} k! \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(\bar{1}, \bar{1}, \{1\}_{k-1})}{n} (-1)^n x^n. \quad (3.17)$$

Hence, combining (3.16) and (3.17), it is easily shown that

$$J_m(k) = (-1)^{m+k} k \zeta(\bar{1}, \{1\}_{m-1}, \bar{1}, \bar{1}, \{1\}_{k-1}). \quad (3.18)$$

Furthermore, using integration by parts we can prove that

$$\begin{aligned} J_m(k) &= (-1)^{m+k-1} k! \sum_{i=1}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, \bar{1}, \bar{1}, \{1\}_{k-1}) \\ &\quad + \frac{(-1)^{m-1}}{(m-1)!} \int_0^1 \frac{\ln^{m-1}(1+t_1)}{1+t_1} dt_1 \int_0^{t_1} \frac{\ln^k(1+t_2)}{1-t_2} dt_2. \end{aligned} \quad (3.19)$$

Moreover, we note that the integral on the right-hand side of (3.19) can be written as

$$\int_0^1 \frac{\ln^{m-1}(1+t_1)}{1+t_1} dt_1 \int_0^{t_1} \frac{\ln^k(1+t_2)}{1-t_2} dt_2$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \left\{ \int_0^x \frac{\ln^{m-1}(1+t_1)}{1+t_1} dt_1 \int_0^{t_1} \frac{\ln^k(1+t_2)}{1-t_2} dt_2 \right\} \\
&= \frac{1}{m} \lim_{x \rightarrow 1} \left\{ \int_0^x \frac{\ln^m(1+x) \ln^k(1+t) - \ln^{m+k}(1+t)}{1-t} dt \right\} \\
&= \frac{1}{m} \left\{ k(\ln 2)^m \int_0^1 \frac{\ln^{k-1}(1+t) \ln(1-t)}{1+t} dt - (m+k) \int_0^1 \frac{\ln^{m+k-1}(1+t) \ln(1-t)}{1+t} dt \right\}. \quad (3.20)
\end{aligned}$$

Therefore, the relations (3.18), (3.19) and (3.20) yield the desired result. The proof of Theorem 3.5 is completed. \square

From Lemma 2.2 and Theorem 3.5, we have the conclusion: if $m, k \in \mathbb{N}$, then the alternating multiple zeta values $\zeta(\bar{1}, \{1\}_{m-1}, \bar{1}, \{1\}_{k-1})$ can be expressed as a rational linear combination of zeta values, polylogarithms and $\ln 2$. We now close this section with several examples.

Example 3.1 By (3.14) and (3.15), we have

$$\begin{aligned}
\zeta(\bar{1}, 1, \bar{1}) &= \frac{1}{8} \zeta(3) - \frac{1}{6} \ln^3 2, \\
\zeta(\bar{1}, \bar{1}, \bar{1}) &= -\frac{1}{4} \zeta(3) + \frac{1}{2} \zeta(2) \ln 2 - \frac{1}{6} \ln^3 2, \\
\zeta(\bar{1}, 1, 1, \bar{1}) &= \text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{12} \ln^4 2 + \frac{7}{8} \zeta(3) \ln 2 - \frac{1}{2} \zeta(2) \ln^2 2 - \zeta(4), \\
\zeta(\bar{1}, \bar{1}, \bar{1}, 1) &= 3\text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{6} \ln^4 2 + \frac{23}{8} \zeta(3) \ln 2 - \zeta(2) \ln^2 2 - 3\zeta(4), \\
\zeta(\bar{1}, 1, \bar{1}, \bar{1}) &= -3\text{Li}_4\left(\frac{1}{2}\right) - \frac{1}{12} \ln^4 2 - \frac{11}{4} \zeta(3) \ln 2 - \frac{3}{4} \zeta(2) \ln^2 2 + 3\zeta(4).
\end{aligned}$$

4 Some evaluation of restricted sum formulas involving multiple zeta values

In [22], we considered the following restricted sum formulas involving multiple zeta values

$$(-1)^m \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, 3, \{1\}_{k-1}) + (-1)^k \sum_{i=0}^{k-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{k-i-1}, 3, \{1\}_{m-1})$$

and gave explicit reductions to multiple zeta values of depth less than $\max\{k+2, m+2\}$, where $m, k \in \mathbb{N}$. Moreover, we also proved that the restricted sum

$$\sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, 2, \{1\}_{k-1})$$

can be expressed by the zeta values and polylogarithms, which implies that for any $m, k \in \mathbb{N}$, the multiple zeta values $\zeta(\bar{1}, \{1\}_{m-1}, 2, \{1\}_{k-1})$ can be represented as a polynomial of zeta values and polylogarithms with rational coefficients. In particular, one can find explicit formula for weight 4

$$\zeta(\bar{1}, 1, 2) = 3\text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{8} \ln^4 2 + \frac{23}{8} \zeta(3) \ln 2 - \zeta(2) \ln^2 2 - 3\zeta(4).$$

In this section, we will consider the general restricted sum

$$(-1)^m \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, p+3, \{1\}_{k-1}) + (-1)^{p+k} \sum_{i=0}^{k-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{k-i-1}, p+3, \{1\}_{m-1})$$

where $m, k \in \mathbb{N}$ and $p \in \mathbb{N}_0$. Now we are ready to state and prove our main results. Note that our proof of Theorem 4.1 is based on Lemma 3.1.

Theorem 4.1 *For integers $m, k \in \mathbb{N}$ and $p \in \mathbb{N}_0$, we have*

$$\begin{aligned} & (-1)^m \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, p+3, \{1\}_{k-1}) \\ & + (-1)^{p+k} \sum_{i=0}^{k-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{k-i-1}, p+3, \{1\}_{m-1}) \\ & = \frac{(-1)^{m+1}}{m!} (\ln 2)^m \zeta(\overline{p+3}, \{1\}_{k-1}) \\ & + \frac{(-1)^{p+k+1}}{k!} (\ln 2)^k \zeta(\overline{p+3}, \{1\}_{m-1}) \\ & + \sum_{i=0}^p (-1)^i \zeta(\overline{2+i}, \{1\}_{m-1}) \zeta(\overline{p+2-i}, \{1\}_{k-1}). \end{aligned} \quad (4.1)$$

Proof. Similarly as in the proof of Theorem 3.5, by considering the multiple integral

$$R_{m,k}(p) := \int_{0 < t_{m+p+2} < \dots < t_1 < 1} \frac{\ln^k(1+t_{m+p+2})}{(1+t_1) \cdots (1+t_m) t_{m+1} \cdots t_{m+p+1} t_{m+p+2}} dt_1 \cdots dt_{m+p+2}.$$

Then with the help of formula (3.1), we may easily deduce that

$$R_{m,k}(p) = (-1)^{m+k} k! \zeta(\bar{1}, \{1\}_{m-1}, p+3, \{1\}_{k-1}). \quad (4.2)$$

On the other hand, using integration by parts, it is easily shown that

$$\begin{aligned} & (-1)^{m+k} k! \zeta(\bar{1}, \{1\}_{m-1}, p+3, \{1\}_{k-1}) \\ & = (-1)^{m+k+1} k! \sum_{i=1}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-1-i}, p+3, \{1\}_{k-1}) \\ & + \frac{(-1)^{m-1}}{(m-1)!} \int_0^1 \frac{\ln^{m-1}(1+t_1)}{1+t_1} dt_1 \int_0^{t_1} \frac{1}{t_2} dt_2 \cdots \int_0^{t_{p+1}} \frac{1}{t_{p+2}} t_{p+2} \int_0^{t_{p+2}} \frac{\ln^k(1+t_{p+3})}{t_{p+3}} dt_{p+3} \\ & = (-1)^{m+k+1} k! \sum_{i=1}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-1-i}, p+3, \{1\}_{k-1}) \\ & + \frac{(-1)^{m+k-1}}{m!} k! (\ln 2)^m \zeta(\overline{p+3}, \{1\}_{k-1}) \\ & + \frac{(-1)^m}{m!} \int_0^1 \frac{\ln^m(1+t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^k(1+t_{p+2})}{t_{p+2}} dt_{p+2}. \end{aligned} \quad (4.3)$$

Hence, by a direct calculation, we can get the following identity

$$\begin{aligned}
& \int_0^1 \frac{\ln^m(1+t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^k(1+t_{p+2})}{t_{p+2}} dt_{p+2} \\
&= (-1)^k m! k! \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-1-i}, p+3, \{1\}_{k-1}) \\
&\quad + (-1)^k k! (\ln 2)^m \zeta(\overline{p+3}, \{1\}_{k-1}).
\end{aligned} \tag{4.4}$$

Moreover, by using integration by parts again, we deduce that

$$\begin{aligned}
& \int_0^1 \frac{\ln^m(1+t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^k(1+t_{p+2})}{t_{p+2}} dt_{p+2} \\
&+ (-1)^p \int_0^1 \frac{\ln^k(1+t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^m(1+t_{p+2})}{t_{p+2}} dt_{p+2} \\
&= (-1)^{m+k} k! m! \sum_{i=0}^p (-1)^i \zeta(\bar{2}+i, \{1\}_{m-1}) \zeta(\overline{p+2-i}, \{1\}_{k-1}).
\end{aligned} \tag{4.5}$$

Thus, combining the formulas (4.4) and (4.5), we obtain the desired result. This completes the proof of Theorem 4.1. \square

Taking $p = 0$ in Theorem 4.1, we get the following corollary which was first proved in [22].

Corollary 4.2 *For positive integers m and k , we have*

$$\begin{aligned}
& (-1)^m \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, 3, \{1\}_{k-1}) \\
&+ (-1)^k \sum_{i=0}^{k-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{k-i-1}, 3, \{1\}_{m-1}) \\
&= \frac{(-1)^{m+1}}{m!} (\ln 2)^m \zeta(\bar{3}, \{1\}_{k-1}) + \frac{(-1)^{k+1}}{k!} (\ln 2)^k \zeta(\bar{3}, \{1\}_{m-1}) \\
&\quad + \zeta(\bar{2}, \{1\}_{m-1}) \zeta(\bar{2}, \{1\}_{k-1}).
\end{aligned} \tag{4.6}$$

Theorem 4.3 *For positive integers m, k and p , then the following identity holds:*

$$\begin{aligned}
& (-1)^{k+p} m! k! \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, 2, \{1\}_{p-1}, 2, \{1\}_{k-1}) \\
&+ (-1)^m k! m! \sum_{i=0}^{k-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{k-i-1}, 2, \{1\}_{p-1}, 2, \{1\}_{m-1}) \\
&+ (-1)^{k+p} k! (\ln 2)^m \zeta(\bar{2}, \{1\}_{p-1}, 2, \{1\}_{k-1}) \\
&+ (-1)^m m! (\ln 2)^k \zeta(\bar{2}, \{1\}_{p-1}, 2, \{1\}_{m-1})
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{m+k+p} m!k! \zeta(\bar{2}, \{1\}_{m-1}) \zeta(\bar{1}, \{1\}_{p-1}, 2, \{1\}_{k-1}) \\
&\quad + (-1)^{m+k} k!m! \zeta(\bar{2}, \{1\}_{k-1}) \zeta(\bar{1}, \{1\}_{p-1}, 2, \{1\}_{m-1}) \\
&\quad + (-1)^{m+k+p} \sum_{i=1}^{p-1} (-1)^i \zeta(\bar{1}, \{1\}_{i-1}, 2, \{1\}_{m-1}) \zeta(\bar{1}, \{1\}_{p-i-1}, 2, \{1\}_{k-1}). \tag{4.7}
\end{aligned}$$

Proof. The proofs' process of Theorem 4.3 is similar to the proof of Theorem 4.1. We consider the iterated integral

$$Q_{m,k}(p) := \int_{0 < t_{m+p+2} < \dots < t_1 < 1} \frac{\ln^k(1+t_{m+p+2}) dt_1 \cdots dt_{m+p+2}}{(1+t_1) \cdots (1+t_m) t_{m+1} (1+t_{m+2}) \cdots (1+t_{m+p+1}) t_{m+p+2}}.$$

By a similar argument as in the proof of formula (4.1), we have the following identities

$$Q_{m,k}(p) = (-1)^{k+m+p} k! \zeta(\bar{1}, \{1\}_{m-1}, 2, \{1\}_{p-1}, 2, \{1\}_{k-1}), \tag{4.8}$$

$$\begin{aligned}
&\int_0^1 \frac{\ln^m(1+t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{1+t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{1+t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^k(1+t_{p+2})}{t_{p+2}} dt_{p+2} \\
&= (-1)^{k+p} m!k! \sum_{i=0}^{m-1} \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i-1}, 2, \{1\}_{p-1}, 2, \{1\}_{k-1}) \\
&\quad + (-1)^{p+k} k! (\ln 2)^m \zeta(\bar{2}, \{1\}_{p-1}, 2, \{1\}_{k-1}), \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \frac{\ln^m(1+t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{1+t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{1+t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^k(1+t_{p+2})}{t_{p+2}} dt_{p+2} \\
&+ (-1)^p \int_0^1 \frac{\ln^k(1+t_1)}{t_1} dt_1 \int_0^{t_1} \frac{1}{1+t_2} dt_2 \cdots \int_0^{t_p} \frac{1}{1+t_{p+1}} t_{p+1} \int_0^{t_{p+1}} \frac{\ln^m(1+t_{p+2})}{t_{p+2}} dt_{p+2} \\
&= (-1)^{m+k+p} m!k! \zeta(\bar{2}, \{1\}_{m-1}) \zeta(\bar{1}, \{1\}_{p-1}, 2, \{1\}_{k-1}) \\
&\quad + (-1)^{m+k} k!m! \zeta(\bar{2}, \{1\}_{k-1}) \zeta(\bar{1}, \{1\}_{p-1}, 2, \{1\}_{m-1}) \\
&\quad + (-1)^{m+k+p} \sum_{i=1}^{p-1} (-1)^i \zeta(\bar{1}, \{1\}_{i-1}, 2, \{1\}_{m-1}) \zeta(\bar{1}, \{1\}_{p-i-1}, 2, \{1\}_{k-1}). \tag{4.10}
\end{aligned}$$

Hence, combining formulas (4.8)-(4.10) we may easily deduce the result. \square

Setting $p = 2, k = m = 1$ in Theorem 4.3, we get

$$\zeta(\bar{1}, 2, 1, 2) + \ln 2 \zeta(\bar{2}, 1, 2) + \zeta(\bar{2}) \zeta(\bar{1}, 1, 2) = 0.$$

In fact, proceeding in a similar method to evaluation of the Theorem 3.2, 4.1 and 4.3, it is possible to evaluate other alternating multiple zeta values. For example, we have used our method to obtain the following explicit integral representations, closed form representations of multiple zeta values:

$$\zeta(\{\bar{1}\}_{2p+2}, \{1\}_{k-1}) = (-1)^{p+1} \text{Li}_{k+1, \{2\}_p} \left(\frac{1}{2} \right)$$

$$= \frac{(-1)^{k+p+1}}{k!} \int_{0 < t_{2p+1} < \dots < t_1 < 1} \frac{\ln^k(1 - t_{2p+1})}{\prod_{j=1}^p \{(1 + t_{2j-1})(1 - t_{2j})\} (1 + t_{2p+1})} dt_1 \dots dt_{2p+1}, \quad (4.11)$$

$$\zeta(\{\bar{1}\}_{2p+1}, \{1\}_{k-1}) = \frac{(-1)^{k+p}}{k!} \int_{0 < t_{2p} < \dots < t_1 < 1} \frac{\ln^k(1 + t_{2p})}{\prod_{j=1}^p \{(1 + t_{2j-1})(1 - t_{2j})\}} dt_1 \dots dt_{2p}, \quad (4.12)$$

$$\sum_{i=0}^m \frac{(\ln 2)^i}{i!} \zeta(\bar{1}, \{1\}_{m-i}, \{\bar{1}\}_{2p}, \{1\}_{k-1}) = \frac{(-1)^{k+p}}{k!m!} \int_{0 < t_{2p} < \dots < t_1 < 0} \frac{\ln^m(1 + t_1) \ln^k(1 + t_{2p})}{\prod_{j=1}^p \{(1 + t_{2j-1})(1 - t_{2j})\}} dt_1 \dots dt_{2p}, \quad (4.13)$$

$$\zeta(\bar{1}, \{1\}_m, \{\bar{1}\}_{2p+1}, \{1\}_{k-1}) = (-1)^{m+p+1} \text{Li}_{k+1, \{2\}_p, \{1\}_m} \left(\frac{1}{2} \right). \quad (4.14)$$

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